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# The negative Toda hierarchy and rational Poisson brackets

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**Abstract**

In this paper we extend the usual hierarchies for the finite, nonperiodic Toda lattice for negative values of the index. We define an infinite sequence of rational homogeneous Poisson brackets, master symmetries, invariants and investigate the various relationships between them. All the relations between master symmetries, Poisson tensors and invariants which hold over the positive integers are extended for all integer values. We comment on extensions to other versions of the Toda lattice, i.e. the periodic, infinite and Bogoyavlensky–Toda type systems.


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## 1. Introduction

The Toda lattice is a Hamiltonian system with Hamiltonian function

$$H(q_1, \dots, q_N, p_1, \dots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}. \tag{1}$$

The function $q_j(t)$ is the position of the $j$th particle and $p_j(t)$ the corresponding momentum. This is the classical, finite, nonperiodic Toda lattice. This system was investigated in [11,12,15,21,24,25,34].

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Hamilton's equations become

$$\dot{q}_j = p_j, \qquad \dot{p}_j = \mathrm{e}^{q_{j-1}-q_j} - \mathrm{e}^{q_j-q_{j+1}}.$$

This system is integrable. One can find a set of independent functions $\{H_1, \dots, H_N\}$ which are constants of motion for Hamilton's equations. To determine the constants of motion, one uses Flaschka's transformation:

$$a_i = \tfrac{1}{2}\mathrm{e}^{1/2(q_i-q_{i+1})}, \qquad b_i = -\tfrac{1}{2}p_i. \tag{2}$$

Then

$$\dot{a}_i = a_i(b_{i+1} - b_i), \qquad \dot{b}_i = 2(a_i^2 - a_{i-1}^2).$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where $L$ is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & & \vdots \\ 0 & -a_2 & 0 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & \cdots & & -a_{N-1} & 0 \end{pmatrix}.$$

This is an example of an isospectral deformation; the entries of $L$ vary over time but the eigenvalues remain constant. It follows that the functions $H_i = (1/i)\,\mathrm{tr}\,L^i$ are constants of motion. We note that

$$H_1 = \sum_{i=1}^{N} b_i = -\frac{1}{2}(p_1 + p_2 + \cdots + p_N)$$

and

$$H_2 = H(q_1, \dots, q_N, p_1, \dots, p_N) = \frac{1}{2}\sum_{i=1}^{N} b_i^2 + \sum_{i=1}^{N-1} a_i^2.$$

Consider $\mathbf{R}^{2N}$ with coordinates $(q_1, \dots, q_N, p_1, \dots, p_N)$, the standard symplectic bracket

$$\{f, g\}_s = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{3}$$

and the mapping $F : \mathbf{R}^{2N} \to \mathbf{R}^{2N-1}$ defined by

$$F : (q_1, \dots, q_N, p_1, \dots, p_N) \to (a_1, \dots, a_{N-1}, b_1, \dots, b_N).$$

Define a bracket on $\mathbf{R}^{2N-1}$ by

$$\{f, g\} = \{f \circ F, g \circ F\}_s.$$

The result is a bracket which (up to a constant multiple) is given by

$$\{a_i, b_i\} = -a_i, \qquad \{a_i, b_{i+1}\} = a_i. \tag{4}$$

All other brackets are zero. $H_1 = b_1 + b_2 + \cdots + b_N$ is the only Casimir. The Hamiltonian in this bracket is $H_2 = 1/2 \operatorname{tr} L^2$. We also have involution of invariants, $\{H_i, H_j\} = 0$. The Lie algebraic interpretation of this bracket can be found in [16]. We denote this bracket by $\pi_1$. The quadratic Toda bracket appears in conjunction with isospectral deformations of Jacobi matrices. First, let $\lambda$ be an eigenvalue of $L$ with normalized eigenvector $v$. Standard perturbation theory shows that

$$\nabla\lambda = (2v_1v_2, \dots, 2v_{N-1}v_N, v_1^2, \dots, v_N^2)^{\mathrm{T}} := U^\lambda,$$

where $\nabla\lambda$ denotes $(\partial\lambda/\partial a_1, \dots, \partial\lambda/\partial b_N)$. Some manipulations show that $U^\lambda$ satisfies

$$\pi_2 U^\lambda = \lambda \pi_1 U^\lambda,$$

where $\pi_1$ and $\pi_2$ are skew-symmetric matrices. It turns out that $\pi_1$ is the matrix of coefficients of the Poisson tensor (4), and $\pi_2$, whose coefficients are quadratic functions of the $a$'s and $b$'s, can be used to define a new Poisson tensor. The quadratic Toda bracket appeared in a paper of Adler [1] in 1979. It is a Poisson bracket in which the Hamiltonian vector field generated by $H_1$ is the same as the Hamiltonian vector field generated by $H_2$ with respect to the $\pi_1$ bracket. The defining relations are

$$\begin{aligned} &\{a_i, a_{i+1}\} = \tfrac{1}{2} a_i a_{i+1}, \qquad \{a_i, b_i\} = -a_i b_i, \\ &\{a_i, b_{i+1}\} = a_i b_{i+1}, \qquad \{b_i, b_{i+1}\} = 2a_i^2. \end{aligned} \tag{5}$$

All other brackets are zero. This bracket has $\det L$ as Casimir and $H_1 = \operatorname{tr} L$ is the Hamiltonian. The eigenvalues of $L$ are still in involution. Furthermore, $\pi_2$ is compatible with $\pi_1$. We also have

$$\pi_2 \nabla H_l = \pi_1 \nabla H_{l+1}.$$

These relations are similar to the Lenard relations for the KdV equation; they are generally called the Lenard relations.

Finally, we remark that further manipulations with the Lenard relations for the infinite Toda lattice, followed by setting all but finitely many $a_i$, $b_i$ equal to zero, yield another

Poisson bracket, $\pi_3$, which is cubic in the coordinates (see [17]). The defining relations for $\pi_3$ are

$$\{a_i, a_{i+1}\} = a_i a_{i+1} b_{i+1}, \qquad \{a_i, b_i\} = -a_i b_i^2 - a_i^3, \qquad \{a_i, b_{i+1}\} = a_i b_{i+1}^2 + a_i^3,$$
$$\{a_i, b_{i+2}\} = a_i a_{i+1}^2, \quad \{a_{i+1}, b_i\} = -a_i^2 a_{i+1}, \quad \{b_i, b_{i+1}\} = 2a_i^2(b_i + b_{i+1}). \qquad (6)$$

All other brackets are zero. The bracket $\pi_3$ is compatible with both $\pi_1$ and $\pi_2$ and the eigenvalues of $L$ are still in involution. The Casimir for this bracket is tr $L^{-1}$.

The multi-Hamiltonian structure of the Toda lattice is well known. The results are usually presented either in the natural $(q, p)$ coordinates or in the more convenient Flaschka coordinates $(a, b)$. In the former case the hierarchy of higher invariants are generated by the use of a recursion operator [8,10]. In the latter case one uses master symmetries as in [3,4]. We have to point out that chronologically every result obtained so far was done first in Flaschka coordinates $(a, b)$ and then transferred through the inverse of Flaschka's transformation to the original $(q, p)$ coordinates. This is to be expected since it is always easier to work with sums of polynomials than with sums of exponentials. In this paper we take the reverse route. We use the recursion operator in $(q, p)$ space to define the various tensorial objects, we then determine the relations which they satisfy, and finally we transfer the results to the more traditional Flaschka coordinates.

The sequence of Poisson tensors can be extended to form an infinite hierarchy. In order to produce the hierarchy of Poisson tensors one uses master symmetries. The first three Poisson brackets are precisely the linear, quadratic and cubic brackets we mentioned above. If a system is bi-Hamiltonian and one of the brackets is symplectic, one can find a recursion operator by inverting the symplectic tensor. The recursion operator is then applied to the initial symplectic bracket to produce an infinite sequence. However, in the case of Toda lattice (in Flaschka variables $(a, b)$) both operators are non-invertible and therefore this method fails. The absence of a recursion operator for the finite Toda lattice is also mentioned in Morosi and Tondo [23] where a Ninjenhuis tensor for the infinite Toda lattice is calculated. Recursion operators were introduced by Olver [30]. Master symmetries were first introduced by Fokas and Fuchssteiner in [13] in connection with the Benjamin–Ono equation. In the case of Toda equations, the master symmetries map invariant functions to other invariant functions. Hamiltonian vector fields are also preserved. New Poisson brackets are generated by using Lie derivatives in the direction of these vector fields and they satisfy interesting deformation relations. We give a summary of the results:

- There exists a sequence of invariants

  $$H_1, H_2, H_3, \ldots,$$

  where $H_i = (1/i)\,\mathrm{tr}\, L^i$.
- A corresponding sequence of Hamiltonian vector fields

  $$\chi_1, \chi_2, \chi_3, \ldots,$$

  where $\chi_i = \chi_{H_i}$.
- A hierarchy of Poisson tensors

  $$\pi_1, \pi_2, \pi_3, \ldots,$$

  where $\pi_i$ is polynomial, homogeneous, of degree $i$.

- Finally, one can determine a sequence of master symmetries

$$X_1, X_2, X_3, \dots,$$

which are used to create the hierarchies through Lie derivatives.

**Note.** Actually, in Ref. [3], one finds a construction of $X_0$ and $X_{-1}$, so $X_i$ is defined for $i \geq -1$.

We quote the results from Refs. [3,4].

**Theorem 1.**

i) $\pi_j$, $j \geq 1$ *are all Poisson.*
ii) *The functions* $H_i$, $i \geq 1$ *are in involution with respect to all* $\pi_j$.
iii) $X_i(H_j) = (i + j)H_{i+j}$, $i \geq -1$, $j \geq 1$.
iv) $L_{X_i}\pi_j = (j - i - 2)\pi_{i+j}$, $i \geq -1$, $j \geq 1$.
v) $[X_i, X_j] = (j - i)X_{i+j}$, $i \geq 0$, $j \geq 0$.
vi) $\pi_j \nabla H_i = \pi_{j-1}\nabla H_{i+1}$, *where* $\pi_j$ *denotes the Poisson matrix of the tensor* $\pi_j$.

To define the vector fields $X_n$ one considers expressions of the form

$$\dot{L} = [B, L] + L^n. \tag{7}$$

This equation is similar to a Lax equation, but in this case the eigenvalues satisfy $\dot{\lambda} = \lambda^{n+1}$ instead of $\dot{\lambda} = 0$ (see [4] for details).

Another approach, which explains these relations is adopted in Das and Okubo [8], and Fernandes [10]. In principle, their method is general and may work for other finite dimensional systems as well. This approach was also used in [26] by da Costa and Marle in the case of the relativistic Toda lattice. The procedure is the following: one defines a second Poisson bracket in the space of canonical variables $(q_1, \dots, q_N, p_1, \dots, p_N)$. This gives rise to a recursion operator. The presence of a conformal symmetry as defined in Oevel [27] allows one, by using the recursion operator, to generate an infinite sequence of master symmetries. These, in turn, project to the space of the new variables $(a, b)$ to produce a sequence of master symmetries in the reduced space. This procedure is described in Section 3.

Section 2 is the background material on master symmetries and a result due to Oevel. Section 3 deals with the positive recursion operator for the Toda lattice in $(q, p)$ coordinates and the various relations between master symmetries, Poisson tensors and invariants. In Section 4 we define the negative recursion operator and develop similar results both in $(q, p)$ and $(a, b)$ coordinates. Section 5 contains some additional results and examples. In Section 6 we comment briefly on possible extensions to other versions of the Toda lattice, i.e. the periodic, infinite and Bogoyavlensky–Toda lattices.

## 2. Oevel's theorem

We assume that the reader is familiar with the concept of Poisson manifold and properties of the Schouten bracket. See for example [18,35,36]. Let $M$ be a $C^\infty$ manifold equipped with two Poisson tensors $J_1$ and $J_2$. The two tensors are called compatible if $J_1 + J_2$ is Poisson. If $J_1$ is symplectic, we call the Poisson pair $(J_1, J_2)$ non-degenerate. In this case, the $(1, 1)$-tensor $\mathcal{R}$ defined by

$$\mathcal{R} = J_2 J_1^{-1} \tag{8}$$

is called the *recursion operator* associated with the non-degenerate pair.

A *bi-Hamiltonian system* is defined by specifying two Hamiltonian functions $h_1$, $h_2$ satisfying

$$J_1 \nabla h_2 = J_2 \nabla h_1,$$

where $J_i$, $i = 1, 2$, denotes the Poisson matrix of the tensor $J_i$.

The theory of bi-Hamiltonian systems was developed by Magri [19]. He established the existence of a hierarchy of mutually commuting functions $h_i$, all in involution with respect to both brackets. They generate mutually commuting bi-Hamiltonian flows $\chi_i$ satisfying the Lenard recursion relations. For more details see [20].

We record, for future reference, the Lenard relations which follow from the results of Magri:

$$J_j \nabla h_i = J_{j-1} \nabla h_{i+1}. \tag{9}$$

For further information on bi-Hamiltonian systems relevant to Toda type systems see [9,14,32,33].

We recall the definition and basic properties of master symmetries. Consider a differential equation on a manifold $M$, defined by a vector field $\chi$. We are mostly interested in the case where $\chi$ is a Hamiltonian vector field. A vector field $Z$ is a *symmetry* of the equation if

$$[Z, \chi] = 0.$$

A vector field $Z$ is a *master symmetry* if

$$[[Z, \chi], \chi] = 0,$$

but

$$[Z, \chi] \neq 0.$$

**Remark.** This definition is perhaps too general. The class of vector fields which fit this description is large. One expects a master symmetry to preserve the hierarchy of invariants and Poisson brackets. This requirement, perhaps, should be part of the definition of master symmetry.

Suppose that we have a bi-Hamiltonian system defined by the Poisson tensors $J_1$, $J_2$ and the Hamiltonians $h_1$, $h_2$. Assume that $J_1$ is symplectic and let $\chi_1 = \chi$. We define the recursion operator $\mathcal{R} = J_2 J_1^{-1}$, the higher flows

$$\chi_i = \mathcal{R}^{i-1} \chi_1$$

and the higher order Poisson tensors

$$J_i = \mathcal{R}^i J_1.$$

For a non-degenerate bi-Hamiltonian system, master symmetries can be generated using a method due to Oevel [27].

**Theorem 2.** *Suppose that $Z_0$ is a conformal symmetry for both $J_1$, $J_2$ and $h_1$, i.e. for some scalars $\lambda$, $\mu$, and $\nu$ we have*

$$\mathcal{L}_{Z_0} J_1 = \lambda J_1, \qquad \mathcal{L}_{Z_0} J_2 = \mu J_2, \qquad \mathcal{L}_{Z_0} h_1 = \nu h_1.$$

*Then the vector fields*

$$Z_i = \mathcal{R}^i Z_0$$

*are master symmetries, the tensors $J_i$ are Poisson, and we have*

(a) $\mathcal{L}_{Z_i} h_j = (\nu + (j-1+i)(\mu-\lambda))h_{i+j}$.
(b) $\mathcal{L}_{Z_i} J_j = (\mu + (j-i-2)(\mu-\lambda))J_{i+j}$.
(c) $[Z_i, Z_j] = (\mu-\lambda)(j-i)Z_{i+j}$.

## 3. The positive Toda hierarchy

Let $\hat{J}_1$ be the symplectic bracket (3) with Poisson matrix

$$\hat{J}_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where $I$ is the $N \times N$ identity matrix. We use $J_1 = 4\hat{J}_1$. With this convention the bracket $J_1$ is mapped precisely onto the bracket $\pi_1$ under the Flaschka transformation (2). We define $\hat{J}_2$ to be the tensor

$$\hat{J}_2 = \begin{pmatrix} A & B \\ -B & C \end{pmatrix},$$

where $A$ is the skew-symmetric matrix defined by $a_{ij} = 1 = -a_{ji}$ for $i < j$, $B$ the diagonal matrix $(-p_1, -p_2, \dots, -p_N)$ and $C$ the skew-symmetric matrix whose non-zero terms are $c_{i,i+1} = -c_{i+1,i} = \mathrm{e}^{q_i - q_{i+1}}$ for $i = 1, 2, \dots, N-1$. We define $J_2 = 2\hat{J}_2$. With this convention the bracket $J_2$ is mapped precisely onto the bracket $\pi_2$ under the Flaschka transformation. It is easy to see that we have a bi-Hamiltonian pair. We define

$$h_1 = -2(p_1 + p_2 + \cdots + p_N)$$

and $h_2$ to be the Hamiltonian:

$$h_2 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} \mathrm{e}^{q_i - q_{i+1}}.$$

Under Flaschka's transformation (2), $h_1$ is mapped onto $4(b_1 + b_2 + \cdots + b_N) = 4\,\mathrm{tr}\,L = 4H_1$ and $h_2$ is mapped onto $2\,\mathrm{tr}\,L^2 = 4H_2$. Using the relationship $\pi_2 \nabla H_1 = \pi_1 \nabla H_2$ which is part (vi) of Theorem 1 we obtain, after multiplication by 4, the following pair:

$$J_1 \nabla h_2 = J_2 \nabla h_1.$$

We define the recursion operator as follows:

$$\mathcal{R} = J_2 J_1^{-1}.$$

The matrix form of $\mathcal{R}$ is quite simple:

$$\mathcal{R} = \frac{1}{2} \begin{pmatrix} B & -A \\ C & B \end{pmatrix}. \tag{10}$$

This operator raises degrees and we therefore call it the *positive Toda operator*. In $(q, p)$ coordinates, the symbol $\chi_i$ is a shorthand for $\chi_{h_i}$. It is generated as usual by

$$\chi_i = \mathcal{R}^{i-1} \chi_1.$$

In a similar fashion we obtain the higher order Poisson tensors

$$J_i = \mathcal{R}^i J_1.$$

We finally define the conformal symmetry

$$Z_0 = \sum_{i=1}^{N} (N - 2i + 1) \frac{\partial}{\partial q_i} + \sum_{i=1}^{N} p_i \frac{\partial}{\partial p_i}.$$

It is straightforward to verify that

$$\mathcal{L}_{Z_0} J_1 = -J_1, \qquad \mathcal{L}_{Z_0} J_2 = 0.$$

In fact, $Z_0$ is Hamiltonian in the $J_2$ bracket with Hamiltonian function $1/2 \sum_{i=1}^{N} q_i$ (see [10]). This observation will be generalized in Section 5.

In addition

$$Z_0(h_1) = h_1, \qquad Z_0(h_2) = 2h_2.$$

Consequently, $Z_0$ is a conformal symmetry for $J_1$, $J_2$ and $h_1$. The constants appearing in Theorem 2 are $\lambda = -1$, $\mu = 0$ and $\nu = 1$. According to Oevel's theorem we end up with the following deformation relations:

$$[Z_i, h_j] = (i + j) h_{i+j}, \qquad L_{Z_i} J_j = (j - i - 2) J_{i+j}, \qquad [Z_i, Z_j] = (j - i) Z_{i+j}.$$

Switching to Flaschka coordinates, we obtain relations (iii)–(v) of Theorem 1.

## 4. The negative Toda hierarchy

To define the negative Toda hierarchy we use the inverse of the positive recursion operator $\mathcal{R}$. We define

$$\mathcal{N} = \mathcal{R}^{-1} = J_1 J_2^{-1}.$$

Obviously we can use the same conformal symmetry $Z_0 = K_0$ and take $\lambda = 0$, $\mu = -1$ and $\nu = 2$. In other words the role of $\lambda$ and $\mu$ is reversed. We define the vector fields

$$K_i = \mathcal{N}^i K_0 = \mathcal{N}^i Z_0, \quad i = 1, 2, \dots,$$

which are master symmetries. We use the convention $Y_{-i} = K_i$ for $i = 0, 1, 2, \dots$. For example, $Y_{-1} = K_1 = \mathcal{N} Z_0 = -2\sum_{i=1}^{N} \partial/\partial p_i$. This vector field, in $(a, b)$ coordinates, is given by

$$X_{-1} = \nabla H_1 = \nabla \operatorname{tr} L = \sum_{i=1}^{N} \frac{\partial}{\partial b_i}.$$

This is precisely the same vector field which appears in [3]. In that paper $X_{-1}$ was constructed through a different method. Similarly, the vector field $Z_0$ corresponds to the Euler vector field

$$X_0 = \sum_{i=1}^{N-1} a_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial b_i}.$$

**Note.** We use the symbol $Y_i$ for a vector field in $(q, p)$ coordinates and $X_i$ for the same vector field in $(a, b)$ coordinates. Similarly, we denote by $J_i$ a Poisson tensor in $(p, q)$ coordinates and $\pi_i$ the corresponding Poisson tensor in $(a, b)$ coordinates. The index $i$ ranges over all integers.

We now calculate, using Oevel's theorem:

$$[Y_{-i}, Y_{-j}] = [K_i, K_j] = (\mu - \lambda)(j - i)K_{i+j} = (-1)(j - i)K_{i+j} = (i - j)Y_{-(i+j)}.$$

Letting $m = -i$ and $n = -j$ we obtain the relationship

$$[Y_m, Y_n] = (n - m)Y_{m+n} \tag{11}$$

for all $m, n$ negative. The same relation holds in Flaschka coordinates. In other words

$$[X_m, X_n] = (n - m)X_{m+n} \quad \forall m, n \in \mathbf{Z}^-.$$

This last relation may be modified to hold for any two arbitrary integers $m, n$. We suppose, without loss of generality, that $j > i$ and consider the bracket of two master symmetries $K_i = Y_{-i}$ and $Z_j = Y_j$, one in the negative hierarchy and the second in the positive hierarchy, i.e.:

$$K_i = \mathcal{N}^i Z_0 = R^{-i} Z_0$$

and

$$Z_j = \mathcal{R}^j Z_0.$$

We proceed as in the proof of Oevel's theorem (see [10]). First we note that

$$\mathcal{L}_{Z_0}\mathcal{R} = (\mathcal{L}_{Z_0}J_2)J_1^{-1} - J_2 J_1^{-1}\mathcal{L}_{Z_0}J_1^{-1} = (\mu - \lambda)\mathcal{R}.$$

On the other hand

$$\mathcal{L}_{Z_0}\mathcal{N} = \mathcal{L}_{Z_0}(J_1 J_2^{-1}) = (\lambda - \mu)\mathcal{N}.$$

Finally

$$\begin{aligned}[Y_{-i}, Y_j] &= [K_i, Z_j] = [\mathcal{N}^i Z_0, \mathcal{R}^j Z_0] = \mathcal{N}^i \mathcal{L}_{Z_0}(\mathcal{R}^j) Z_0 - \mathcal{R}^j \mathcal{L}_{Z_0}(\mathcal{N}^i) Z_0 \\ &= \mathcal{N}^i j(\mu - \lambda)\mathcal{R}^j Z_0 - \mathcal{R}^j i(\lambda - \mu)\mathcal{N}^i Z_0 \\ &= j(\mu - \lambda)\mathcal{R}^{j-i} Z_0 - i(\lambda - \mu)\mathcal{R}^{j-i} Z_0 \\ &= (i + j)(\mu - \lambda)\mathcal{R}^{j-i} Z_0 = (i + j)(\mu - \lambda) Y_{j-i}.\end{aligned}$$

In the case of Toda lattice $\mu = 0$ and $\lambda = -1$, therefore

$$[Y_{-i}, Y_j] = (i + j) Y_{j-i}.$$

We deduce that (11) holds for any integer value of the index.

We define $W_i = J_{3-i}$. This is necessary since the conclusions of Oevel's theorem assume that the index begins at $i = 1$ and is positive. We compute

$$\begin{aligned}\mathcal{L}_{Y_{-i}} J_{-j} &= \mathcal{L}_{K_i} W_{j+3} = (\mu + (j + 3 - 2 - i)(\mu - \lambda)) W_{i+j+3} \\ &= (i - j - 2) W_{i+j+3} = (i - j - 2) J_{-(i+j)}.\end{aligned}$$

Letting $m = -i$ and $n = -j$ we obtain

$$L_{Y_m} J_n = (n - m - 2) J_{n+m}$$

for $n, m$ negative integers. Switching to Flaschka coordinates we deduce that the relation (iv) of Theorem 1 holds also for negative values of the index. In other words

$$L_{X_i}\pi_j = (j - i - 2)\pi_{i+j}, \quad i \le 0, \quad j \le 0.$$

Again, a straightforward modification of the proof of Oevel's theorem shows that the last relationship holds for any integer value of $m, n$. We have shown that conclusions (iv) and (v) of Theorem 1 hold for integer values of the index. In fact, it is not difficult to demonstrate all the other parts of Theorem 1.

**Theorem 3.** *The conclusions of Theorem 1 hold for any integer value of the index.*

**Proof.** We need to prove parts (i)–(iii) and (vi) of the theorem (not necessarily in that order).

(i) The fact that $J_n$ are Poisson for $n \in \mathbf{Z}$ follows from properties of the recursion operator and it is also part of Oevel's theorem. The similar result in $(a, b)$ coordinates follows easily from properties of the Schouten bracket, and the fact that $J_n$ and $\pi_n$ are $F$-related. We have $\pi_n = F_* J_n$, therefore

$$[\pi_n, \pi_n] = [F_*(J_n), F_*(J_n)] = F_*[J_n, J_n] = F_*(0) = 0.$$

The vanishing of the Schouten bracket is equivalent to the Poisson property. (iii) The case where $i$ and $j$ are both of the same sign was already proved. We next note that $X_n(\lambda) = \lambda^{n+1}$ if $\lambda$ is an eigenvalue of $L$. This follows from Eq. (7) which is used to define the vector fields $X_n$ for $n \geq 0$. We would like to extend the formula $X_n(\lambda) = \lambda^{n+1}$ for $n < 0$. Since $X_{-1}(\lambda) = 1$ we consider $X_{-2}$. We look at the equation

$$[X_{-2}, X_n] = (n+2)X_{n-2}.$$

We act on $\lambda$ with both sides of the equation and let $X_{-2}(\lambda) = f(\lambda)$. We obtain the equation

$$(n+1)\lambda f(\lambda) - f'(\lambda)\lambda^2 = (n+2).$$

This is a linear first-order ordinary differential equation with general solution

$$f(\lambda) = \frac{1}{\lambda} + c\lambda^{n+1}.$$

Since $n$ is arbitrary, we obtain $f(\lambda) = 1/\lambda$. In order to calculate $X_{-3}(\lambda)$ we use

$$X_{-3} = -[X_{-1}, X_{-2}].$$

We obtain

$$X_{-3}(\lambda) = X_{-2}X_{-1}(\lambda) - X_{-1}X_{-2}(\lambda) = -X_{-1}\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda^2}.$$

The result follows by induction.

Finally we calculate

$$\begin{aligned} X_i(H_j) &= \frac{1}{j}X_i\left(\sum \lambda_k^j\right) = \frac{1}{j}\left(\sum X_i \lambda_k^j\right) \\ &= \sum \lambda_k^{j-1} X_i(\lambda_k) = \sum \lambda_k^{j-1}\lambda_k^{i+1} = \sum \lambda_k^{i+j} = (i+j)H_{i+j}. \end{aligned}$$

(vi) First we note that $\pi_j \nabla H_i = \pi_{j-1}\nabla H_{i+1}$, holds for $i, j$ of the same sign. More generally, in the positive (or the negative) hierarchy we have the Lenard relations for the eigenvalues, i.e.:

$$\pi_j \nabla\lambda_i = \lambda_i \pi_{j-1}\nabla\lambda_i. \tag{12}$$

Assume now that $i < 0$, $j > 0$. The calculation is straightforward:

$$\pi_j \nabla \frac{1}{i}\sum \lambda_k^i = \sum \lambda_k^{i-1}\pi_j \nabla\lambda_i = \sum \lambda_i^l \pi_{j-1}\nabla\lambda_k = \pi_{j-1}\nabla\frac{1}{i+1}\sum \lambda_k^{i+1}.$$

Therefore

$$\pi_j \nabla H_i = \pi_{j-1}\nabla H_{i+1}. \tag{13}$$

In the case $i > 0$ and $j < 0$ we use exactly the same calculation but use (12) for the negative hierarchy.

(ii) It is clearly enough to show the involution of the eigenvalues of $L$ since $H_i$ are functions of the eigenvalues. We prove involution of eigenvalues by using the Lenard relations (13). We give the proof for the case of the bracket $\pi_j$ with $j > 0$ but if $j < 0$ the proof is identical. First we show that the eigenvalues are in involution with respect to the bracket $\pi_1$. Let $\lambda$ and $\mu$ be two distinct eigenvalues and let $U, V$ be the gradients of $\lambda$ and $\mu$, respectively. We use the notation $\{,\}$ to denote the bracket $\pi_1$ and $\langle\, ,\rangle$ the standard inner product. The Lenard relations (12) translate into $\pi_2 U = \lambda\pi_1 U$ and $\pi_2 V = \mu\pi_1 V$. Therefore

$$\begin{aligned}\{\lambda, \mu\} &= \langle \pi_1 U, V\rangle = \frac{1}{\lambda}\langle \pi_2 U, V\rangle = -\frac{1}{\lambda}\langle U, \pi_2 V\rangle = -\frac{1}{\lambda}\langle U, \mu\pi_1 V\rangle \\ &= -\frac{\mu}{\lambda}\langle U, \pi_1 V\rangle = \frac{\mu}{\lambda}\langle \pi_1 U, V\rangle = \frac{\mu}{\lambda}\{\lambda, \mu\}.\end{aligned}$$

Therefore, $\{\lambda, \mu\} = 0$. To show the involution with respect to all brackets $\pi_j$, and in view of part (iv) of Theorem 1, it is enough to show the following. Let $f_1, f_2$ be two functions in involution with respect to the Poisson bracket $\pi$, let $X$ be a vector field such that $X(f_i) = f_i^2$ for $i = 1, 2$. Define a Poisson bracket $w$ by $w = \mathcal{L}_X\pi$. Then the functions $f_1, f_2$ remain in involution with respect to the bracket $w$. The proof follows trivially if we write $w = \mathcal{L}_X\pi$ in Poisson form:

$$\{f_1, f_2\}_w = X\{f_1, f_2\}_\pi - \{f_1, X(f_2)\}_\pi - \{X(f_1), f_2\}_\pi. \qquad \square$$

**Remark.** We should point out that

$$H_n = \frac{1}{n}\,\mathrm{tr}\, L^n$$

makes sense for $n \neq 0$ but it is undefined for $n = 0$. The reader should interpret the formulas involving $H_0$ as a degenerate case, i.e. $H_0 = \mathrm{tr}\, L^0/0 = N/0 = \infty$. Therefore, the result of $X_{-n}(H_n) = N$ where $N$ is the size of $L$. It makes sense to define

$$X_m(H_0) = \lim_{n\to 0}\frac{1}{n}X_m(\mathrm{tr}\, L^n).$$

For example, $X_{-1}(H_0)$ is calculated by $X_{-1}((1/n)\,\mathrm{tr}\, L^n) = \mathrm{tr}\, L^{n-1}$. Taking the limit as $n \to 0$ gives $X_{-1}(H_0) = \mathrm{tr}\, L^{-1} = -H_{-1}$ which is the correct answer.

## 5. Further results and examples

In this last section we prove some further results and give some specific examples. In Section 3 we noticed that $Z_0$ is Hamiltonian with respect to the $J_2$ bracket with Hamiltonian function $f = 1/2\sum_{i=1}^{N} q_i$. This observation is due to Fernandes [10]. We generalize the result as follows.

**Theorem 4.** *The master symmetry $Y_n, n \in \mathbf{Z}$ is the Hamiltonian vector field of f with respect to the $J_{n+2}$ bracket.*

**Proof.** We will prove the result for the positive hierarchy $Z_n = Y_n$ but the proof for $Y_{-n} = K_n$ is similar. As a first step we show that

$$Z_n(f) = 0 \quad \forall n \geq 0.$$

We recall that

$$Z_0 = \sum_{i=1}^{N}(N-2i+1)\frac{\partial}{\partial q_i} + \sum_{i=1}^{N} p_i \frac{\partial}{\partial p_i}.$$

Since

$$\sum_{i=1}^{N}(N+1-2i) = 0,$$

we obtain

$$Z_0(f) = \frac{1}{2}Z_0\left(\sum_{i=1}^{N} q_i\right) = \frac{1}{2}\left(\sum_{i=1}^{N} Z_0(q_i)\right) = 0.$$

By examining the form (10) of the recursion operator $\mathcal{R}$ we deduce easily that the $q_i$ component of $Z_1$ is

$$Z_1(q_i) = -\frac{1}{2}\left[(N-2i+1)p_i + \sum_{j>i} p_j - \sum_{j<i} p_j\right].$$

In other words, the vector

$$(Z_1(q_1), \ldots, Z_1(q_N))$$

is the product *AP* where

$$A = -\frac{1}{2}\begin{pmatrix} Z_0(q_1) & 1 & 1 & \cdots & \cdots & 1 \\ -1 & Z_0(q_2) & 1 & \cdots & \cdots & \vdots \\ -1 & -1 & Z_0(q_3) & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & 1 \\ -1 & \cdots & & \cdots & -1 & Z_0(q_N) \end{pmatrix}$$

and $P$ the column vector $(p_1, p_2, \ldots, p_N)^t$. Note that $\sum_{i=1}^{N} a_{ij} = 0$ and $\sum_{j=1}^{N} a_{ij} = -Z_0(q_i)$. Therefore

$$Z_1(f) = \frac{1}{2}(Z_1(q_1) + \cdots + Z_1(q_N)) = \frac{1}{2}\sum_{i,j} a_{ij}p_j = \frac{1}{2}\left(\sum_j \left(\sum_i a_{ij}\right) p_j\right) = 0.$$

In the same fashion one proves that $Z_2(f) = 0$. For $n > 2$, we proceed by induction:

$$Z_n = \frac{1}{n-2}[Z_1, Z_{n-1}].$$

Therefore

$$Z_n(f) = \frac{1}{n-2}[Z_1, Z_{n-1}] = \frac{1}{n-2}(Z_1 Z_{n-1} f - Z_{n-1} Z_1 f) = 0$$

by the induction hypothesis.

To complete the proof of the theorem, it is enough to show

$$Z_n = [J_{n+2}, f],$$

where [, ] denotes the Schouten bracket (see [4, pp. 5514–5515]).

First we note that

$$[[J_{n+1}, f], Z_1] + [[f, Z_1], J_{n+1}] + [[Z_1, J_{n+1}], f] = 0$$

due to the super Jacobi identity for the Schouten bracket. Since

$$[Z_1, f] = Z_1(f) = 0,$$

the middle term in the last identity is zero. We obtain

$$[Z_1, [J_{n+1}, f]] = [[Z_1, J_{n+1}], f].$$

Finally, we calculate using induction:

$$\begin{aligned} Z_n &= \frac{1}{n-2}[Z_1, Z_{n-1}] = \frac{1}{n-2}[Z_1, [J_{n+1}, f]] = \frac{1}{n-2}[[Z_1, J_{n+1}], f] \\ &= \frac{1}{n-2}([(n-2)J_{n+2}, f]) = [J_{n+2}, f]. \end{aligned}$$ □

The result of the theorem is striking. It shows that the master symmetries are determined once the Poisson hierarchy is constructed. Of course one requires knowledge of the function $f$. The function $f$ may be constructed by using Noether's theorem: one of the point symmetries of the Toda lattice (see [6, p. 227]) is given by

$$t\left(\sum_{i=1}^{N} \frac{\partial}{\partial q_i}\right).$$

A corresponding time dependent integral produced from Noether's theorem is

$$I = \frac{1}{2}\sum_{i=1}^{N} q_i - \frac{1}{2}t\sum_{i=1}^{N} p_i = f + \frac{1}{4}th_1.$$

Motivated by the results of [5,31], it makes sense to consider the time independent part of $I$ which is precisely the function $f$. It is an interesting question whether this procedure works for other integrable systems as well. We also remark that the integrals are also determined from the knowledge of the Poisson brackets and the function $f$. For example, it follows easily from Theorem 4 that

$$h_{i+1} = \frac{1}{i+1}\{h_i, f\}_3,$$

where $\{,\}_3$ denotes the cubic Toda bracket.

The rational brackets in $(q, p)$ coordinates are given by complicated expressions that are quite hard to write in explicit form. When projected in the space of $(a, b)$ variables they give rational brackets whose numerator is polynomial and the denominator is the determinant of the Jacobi matrix. We give examples of these brackets and master symmetries for $N = 3$.

For example, the tensor $J_0$ is a homogeneous rational bracket of degree 0. It is defined by

$$J_0 = \mathcal{N} J_1 = J_1 J_2^{-1} J_1.$$

In the case of three particles the corresponding bracket $\pi_0$ is given as follows: first define the skew-symmetric matrix $A$ by

$$\begin{aligned} &a_{12} = -\tfrac{1}{2} a_1 a_2 (b_3 + b_1 - b_2), \quad a_{13} = a_1(a_2^2 - b_2 b_3), \quad a_{14} = -a_1(a_2^2 - b_1 b_3),\\ &a_{15} = a_1 a_2^2, \qquad a_{23} = -a_1^2 a_2, \qquad a_{24} = a_2(a_1^2 - b_1 b_3),\\ &a_{25} = -a_2(a_1^2 - b_1 b_2), \qquad a_{34} = -2a_1^2 b_3, \qquad a_{35} = 0, \qquad a_{45} = -2a_2^2 b_1. \end{aligned}$$

The matrix of the tensor $\pi_0$ is defined by

$$\pi_0 = \frac{1}{\det L} A, \tag{14}$$

where $\det L = b_1 b_2 b_3 - a_2^2 b_1 - a_1^2 b_3$. This formula defines a Poisson bracket with one single Casimir $H_2 = 1/2 \operatorname{tr} L^2$. The bracket is defined on the open dense set $\det L \neq 0$. The explicit formulas for the vector fields $X_1$ and $X_2$ are given in [4]; therefore, we will give an example for the vector field $X_{-2}$. In the case $N = 3$ it is given by

$$X_{-2} = \frac{1}{\det L}\left(\sum_{i=1}^{2} r_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{3} s_i \frac{\partial}{\partial b_i}\right),$$

where

$$\begin{aligned} &r_1 = \tfrac{1}{2} a_1 (b_1 - b_2 - 2b_3), \qquad r_2 = \tfrac{1}{2} a_2 (b_3 - 2b_1 - b_2), \qquad s_1 = b_2 b_3 - a_1^2 - a_2^2,\\ &s_2 = b_1 b_3 + a_1^2 + a_2^2, \qquad s_3 = b_1 b_2 - a_1^2 - a_2^2. \end{aligned}$$

We close by considering the Casimirs of these new Poisson brackets.

**Theorem 5.** *The Casimir of* $\pi_n$ *in the open dense set* $\det L \neq 0$ *is* $\operatorname{tr} L^{2-n}$ *for all* $n \neq 2$. *The Casimir of* $\pi_2$ *is* $\det L$.

**Proof.** For $n \geq 1$ the result was proved in [4, Proposition 5, p. 5525]. Therefore, we only have to show that the Casimir of $\pi_{-m}$ is $\operatorname{tr} L^{m+2}$ $(m \geq 0)$. This follows from (9) and the fact that $H_1 = \operatorname{tr} L$ is the Casimir for the Lie–Poisson bracket $\pi_1$:

$$0 = \pi_1 \nabla H_1 = \pi_0 \nabla H_2 = \pi_{-1} \nabla H_3 = \cdots . \qquad \square$$

## 6. Extensions to other versions of the Toda lattice

In this paper we have studied the negative hierarchy for the finite nonperiodic Toda lattice. We would like to close with some remarks on the other versions of the Toda lattice, i.e. the periodic Toda, the infinite Toda with boundary conditions and the Bogoyavlensky–Toda lattice, corresponding to simple Lie groups.

The results may be extended to the case of the periodic Toda lattice. We have to stress that again there is no recursion operator in Flaschka variables $(a, b)$; one has to work in natural $(p, q)$ coordinates and then project. Here are the main differences from the finite nonperiodic Toda lattice. The Hamiltonian now becomes

$$h_2 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N} \mathrm{e}^{q_i - q_{i+1}}.$$

Note that we have added an additional term

$$\mathrm{e}^{q_N - q_{N+1}} = \mathrm{e}^{q_N - q_1}$$

and we assume periodic conditions $q_{N+i} = q_i$ and $p_{N+i} = p_i$. The tensor $J_1$ is the same as in Section 3 but $J_2$ has to be modified. The matrix $\hat{J}_2$ should be replaced with

$$\hat{J}_2 = \begin{pmatrix} A & B \\ -B & C \end{pmatrix},$$

where $A$ and $B$ are the same as before, but $C$ is the skew-symmetric matrix whose non-zero terms are $c_{i,i+1} = -c_{i+1,i} = \mathrm{e}^{q_i - q_{i+1}}$ for $i = 1, 2, \ldots, N-1$ and $c_{1,N} = -c_{N,1} = -\mathrm{e}^{q_N - q_1}$. We define $\mathcal{R}$ and $\mathcal{N}$ as before:

$$\mathcal{R} = J_2 J_1^{-1}, \qquad \mathcal{N} = J_1 J_2^{-1}.$$

We obtain similar results as in the nonperiodic Toda lattice and these in turn project into the space of $(a, b)$ variables. Flaschka's transformation is given by the same formula but with one extra variable

$$a_N = \tfrac{1}{2} \mathrm{e}^{(1/2)(q_N - q_1)}.$$

The Lax pair with a spectral parameter can be found in [2,22] where the periodic Toda lattice is studied in detail. The matrix $L$ is given by

$$L=\begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & \frac{1}{\lambda}a_N \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ \lambda a_N & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix}.$$

The linear and quadratic brackets (4) and (5) are given by the same formula but we take into account the periodicity. For example, in $\pi_1$

$$\{a_N, b_1\} = \{a_N, b_{N+1}\} = a_N.$$

These two tensors are just the beginning of a double hierarchy of Poisson tensors. We have also master symmetries and invariants. We remark that the Poisson tensor $\pi_n$ now has an additional Casimir $C = a_1a_2, \dots, a_N$.

We give an example of one such rational bracket for $N = 3$. We will explicitly compute the formula for $\pi_0$ as we did in the case of the nonperiodic Toda.

First, define the matrix $A$ in block form:

$$A=\begin{pmatrix} A_1 & A_2 \\ -A_2^t & A_3, \end{pmatrix},$$

where

$$A_1=\begin{pmatrix} 0 & -\frac{1}{2}a_1a_2(b_3+b_1-b_2) & \frac{1}{2}a_1a_3(b_3+b_1-b_2) \\ \frac{1}{2}a_1a_2(b_3+b_1-b_2) & 0 & -\frac{1}{2}a_2a_3(b_3+b_1-b_2) \\ -\frac{1}{2}a_1a_3(b_3+b_1-b_2) & \frac{1}{2}a_2a_3(b_3+b_1-b_2) & 0 \end{pmatrix},$$

$$A_2=\begin{pmatrix} a_1(a_2^2-a_3^2-b_2b_3) & -a_1(a_2^2-a_3^2-b_1b_3) & a_1(a_2^2-a_3^2) \\ -a_2(a_1^2-a_3^2) & a_2(a_1^2-a_3^2-b_1b_3) & -a_2(a_1^2-a_3^2-b_1b_2) \\ a_3(a_1^2-a_2^2+b_2b_3) & -a_3(a_1^2-a_2^2) & a_3(a_1^2-a_2^2-b_1b_2) \end{pmatrix}$$

and

$$A_3=\begin{pmatrix} 0 & -2a_1^2b_3 & 2a_3^2b_2 \\ 2a_1^2b_3 & 0 & -2a_2^2b_1 \\ -2a_3^2b_2 & 2a_2^2b_1 & 0 \end{pmatrix}.$$

The matrix of the tensor $\pi_0$ is defined by

$$\pi_0 = \frac{1}{d}A,$$

where $d = b_1b_2b_3 - a_2^2b_1 - a_1^2b_3 + a_3^2b_2$. This formula defines a Poisson bracket with two Casimirs $H_2 = 1/2\,\mathrm{tr}\,L^2$ and $C = a_1a_2a_3$. It is no surprise that setting $a_3 = 0$ gives precisely formula (14) of the nonperiodic case.

Similar results hold in the case of the infinite Toda lattice with boundary conditions. In fact, the only reference we were able to find in the literature on the subject of negative hierarchies concerns this particular case of the relativistic Toda lattice (see [28,29] for explicit calculations).

Finally, we would like to mention an important application of these results in the case of the generalized Toda lattices of Bogoyavlensky. In fact, the work of this paper has originated from an effort to find a bi-Hamiltonian formulation of these systems. For example, in the case of the Bogoyavlensky–Toda lattice of type $B_n$ there exists a recursion operator which gives rise to an infinite hierarchy

$$\pi_1, \pi_3, \pi_5, \ldots$$

of polynomial brackets of odd degree. The Lenard relations begin at the second flow, i.e.:

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4.$$

Using the rational bracket $\pi_{-1}$ we can establish for the first time the bi-Hamiltonian nature of this system, i.e.:

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4.$$

We will report on this development in the future (see [7] for some preliminary work).

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